LIMPOPO MATHS AND SCIENCE ACADEMY

LIMSA

GRADE 12

MATHEMATICS

TRIGONOMETRY

WORKSHEET

DURATION: 7 DAYS **23 TO 29 MARCH 2020**

COMPILED BY (Mr Musemburi)

TRIG GRAPHS

Grade 11 Recap

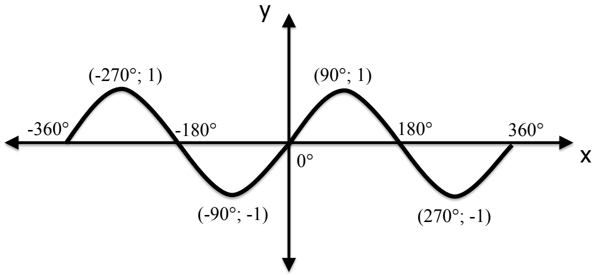
IMPORTANT!

When sketching trig graphs, you need to label the following:

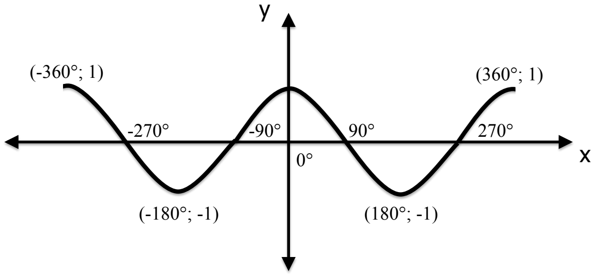
- **both axes**
- **x- and y-intercepts**
- **turning points**
- **endpoints** (if not on the axes)
- **asymptotes** (tan graph only)

BASICS

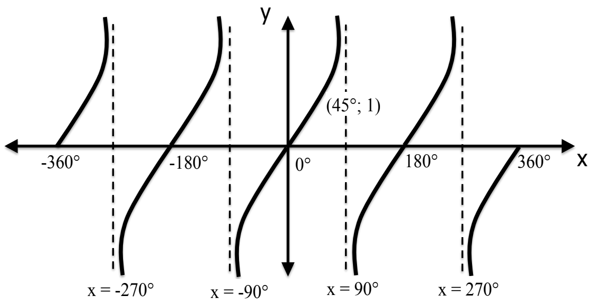
- $y = \sin x$ for $x \in [-360°; 360°]$



- $y = \cos x$ for $x \in [-360°; 360°]$



- $y = \tan x$ for $x \in [-360°; 360°]$



Notes for $\sin x$ and $\cos x$:

❖ Key points (intercepts/turning pts) every 90°

❖ Period (1 complete graph): 360°

❖ Amplitude (halfway between min and max): 1

Notes for $\tan x$:

❖ Key points every 45°

❖ Period (1 complete graph): 180°

❖ No amplitude can be defined

❖ Asymptotes at $x = 90° + k180°, k \in \mathbb{Z}$

VERTICAL SHIFT

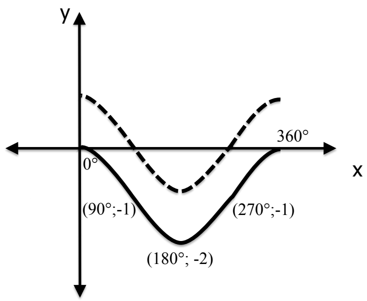
- $y = \sin x + q$ or $y = \cos x + q$ or $y = \tan x + q$

If q > 0 : upwards (e.g: $y = \sin x + 1$)
If q < 0 : downwards (e.g: $y = \cos x - 2$)

EXAMPLE

$y = \cos x - 1 \quad x \in [0°; 360°]$ (solid line)

$y = \cos x$ (dotted line - for comparison)



AMPLITUDE CHANGE

- $y = a\,.\sin x$ or $y = a\,.\cos x$ or $y = a\,.\tan x$

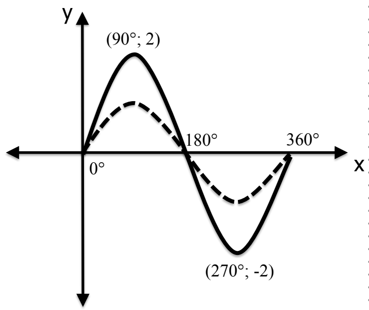
If a > 1 : stretch upwards
0 < a < 1 : compress downward
a < 0 : reflection in x-axis

EXAMPLES

1. $y = 2\sin x$ (solid line)

$y = \sin x$ (dotted line - for comparison)

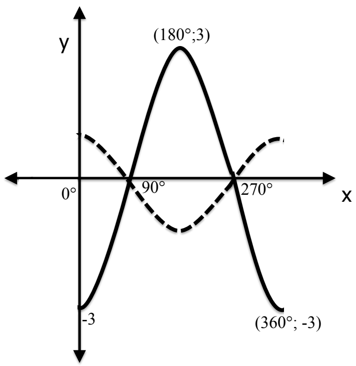
* Amplitude = 2



2. $y = -3\cos x$ (solid line)

$y = \cos x$ (dotted line - for comparison)

* Range: $y \in [-3; 3]$



PERIOD CHANGE

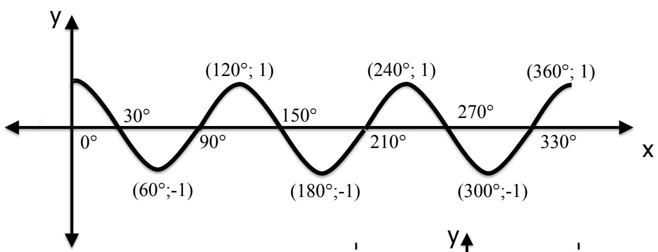
- $y = \sin bx$ or $y = \cos bx$ or $y = \tan bx$

The value of b indicates how many graphs are completed in the 'regular' period of that graph (i.e. $\sin x/\cos x$: 360° and $\tan x$: 180°)

EXAMPLES

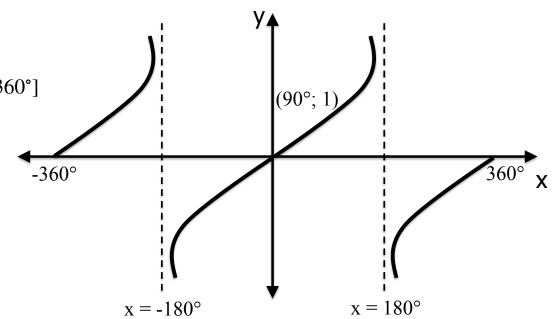
1. $y = \cos 3x \quad x \in [0°; 360°]$

* Normal period: 360°
* New period: 120° (3 graphs in 360°)
* Critical points every 90/3 = **30°**



2. $y = \tan \frac{1}{2}x \quad x \in [0°; 360°]$

* Normal period: 180°
* New period: 360° (½ graph in 180°)
* Critical points: every 45/0,5 = **90°**



HORIZONTAL SHIFT

- $y = \sin(x - p)$ or $y = \cos(x - p)$ or $y = \tan(x - p)$

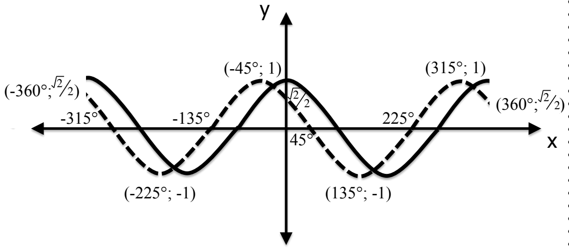
If p > 0 : shift right (e.g: $y = \sin(x - 30°)$)
p < 0 : shift left (e.g: $y = \cos(x + 45)$)

How to plot a horizontal shift:

- Plot the original curve
- Move the critical points left/right
- Label the x-cuts and turning points
- Calculate and label the endpoints and y-cut

EXAMPLES

1. $y = \cos(x + 45°)$ for $x \in [-360°; 360°]$ (dotted line)
$y = \cos x$ (solid line - for comparison)



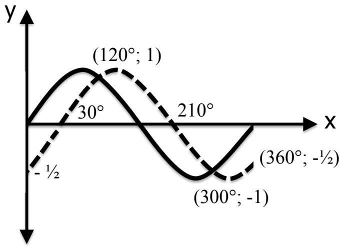
Endpoints:

$\cos(-360° + 45°) = \frac{\sqrt{2}}{2}$ and $cos(-360° + 45°) = \frac{\sqrt{2}}{2}$

y-cut:

$\cos(0° + 45°) = \frac{\sqrt{2}}{2}$

2. $y = \sin(x - 30°)$ for $x \in [0°; 360°]$ (dotted line)
$y = \sin x$ (solid line - for comparison)



Endpoints:

$\sin(0° + 45°) = -\frac{1}{2}$ and $\sin(360° - 30°) = -\frac{1}{2}$

y-cut:

The y-cut is one of the endpoints

EXAMPLE

Given $f(x) = \cos(x + 60°)$ and $g(x) = \sin 2x$

Questions:

1. Determine algebraically the points of intersection of $f(x)$ and $g(x)$ for $x \in$ [-90°;180°]
2. Sketch $f(x)$ and $g(x)$ for $x \in$ [-90°;180°]
3. State the amplitude of $f(x)$
4. Give the period of $g(x)$
5. Use the graphs to determine the values of x for which:
   a. $g(x)$ is increasing and positive
   b. $f(x)$ is increasing and positive
   c. $f(x) \geq g(x)$ - i.e. $f(x)$ is above $g(x)$
   d. $f(x) \cdot g(x) \geq 0$ - i.e. product is + or 0
6. Explain the transformation that takes $y = sin x$ to $y = \sin(2x - 60°)$

Solutions:

1. $\cos(x + 60°) = \sin 2x$

$\cos(x + 60°) = \cos(90° - 2x)$

Reference $\angle$: $90° - 2x$

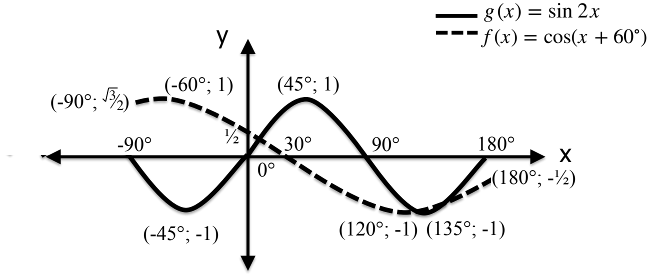
QI: $x + 60° = 90° - 2x + k360°;\ k \in \mathbb{Z}$
$3x = 30° + k360°$
$x = 10° + k120°$

QIV: $x + 60° = 360° - (90° - 2x) + k360°;\ k \in \mathbb{Z}$
$x + 60° = 270° + 2x + k360°$
$-x = 210° + k360°$
$x = -210° + k360°$

but $x \in [-90°; 180°]$

$\therefore\ x = 10°;\ 130°;\ 150°$

2.



For f(x):

Endpoints: $\cos(-90° + 60°) = \frac{\sqrt{3}}{2}$ and $\cos(180° + 60°) = -\frac{1}{2}$

y-cut: $\cos(0° + 60°) = \frac{1}{2}$

3. 1

4. 180°

5. a. $x \in (0°; 45°)$
   b. $x \in [-90°; -60°)$
   c. $x \in [-90°; 10°] \cup (130°; 150°)$
   d. $x \in [0°; 30°] \cup [90°; 180°]$ also at $x = -90°$

6. Rewrite $y = \sin(2x - 60°)$ in the form $y = \sin b(x - p) = \sin(2(x - 30°))$
Transformation: b = 2 ∴ period is halved
p = 30 ∴ shifted 30 to the right°

TRIG GRAPHS

USING TRIG GRAPHS TO FIND RESTRICTIONS ON IDENTITIES

i.e. answering the question
"for which values of x will this identity be undefined?"

Identities are undefined if:

- the function is undefined
  $\tan x$ has asymptotes at $x = 90° + k180°;\ k \in \mathbb{Z}$
- any denominator is zero

Reminder:
$\frac{A}{0}$ is undefined

EXAMPLES

1. For which values of x will $\cos^2 x \cdot \tan^2 x = \sin^2 x$ be <u>defined</u>?

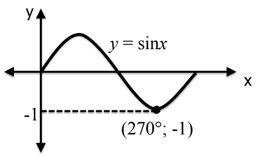
- $\tan x$ is undefined at $x = 90° + k180°;\ k \in \mathbb{Z}$
  $\therefore$ will be defined at $x \in \mathbb{R}$ and $x \neq 90° + k180°;\ k \in \mathbb{Z}$
- no denominators that could be zero

2. For which values of x will $\tan x + \frac{\cos x}{1 + \sin x} = \frac{1}{\cos x}$ be undefined?

- $\tan x$ is undefined at $x = 90° + k180°;\ k \in \mathbb{Z}$
- fractions are undefined if the denominator = 0

$\therefore$ if $1 + \sin x = 0$ or if $\cos x = 0$

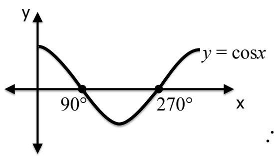
* $1 = \sin x = 0$
$\therefore \sin x = -1$



Use trig graphs for $0; \pm 1$

$\therefore x = 270° + k360°;\ k \in \mathbb{Z}$

* $\cos x = 0$



Use trig graphs for $0; \pm 1$

$\therefore x = 90° + k180°;\ k \in \mathbb{Z}$

$x = 90° + k180°;\ k \in \mathbb{Z}$
$x = 270° + k360°;\ k \in \mathbb{Z}$
$x = 90° + k180°;\ k \in \mathbb{Z}$

can be summarised as: $x = 90° + k180°;\ k \in \mathbb{Z}$

TRIGONOMETRY

CAST DIAGRAM

$180° - \theta$

1. $\sin(180° - \theta) = \sin\theta$
2. $\cos(180° - \theta) = -\cos\theta$
3. $\tan(180° - \theta) = -\tan\theta$

$\theta + 360°$

1. $\sin(\theta + 360°) = \sin\theta$
2. $\cos(\theta + 360°) = \cos\theta$
3. $\tan(\theta + 360°) = \tan\theta$

$90° + \theta$

1. $\sin(90° + \theta) = \cos\theta$
2. $\cos(90° + \theta) = -\sin\theta$

$90° - \theta$

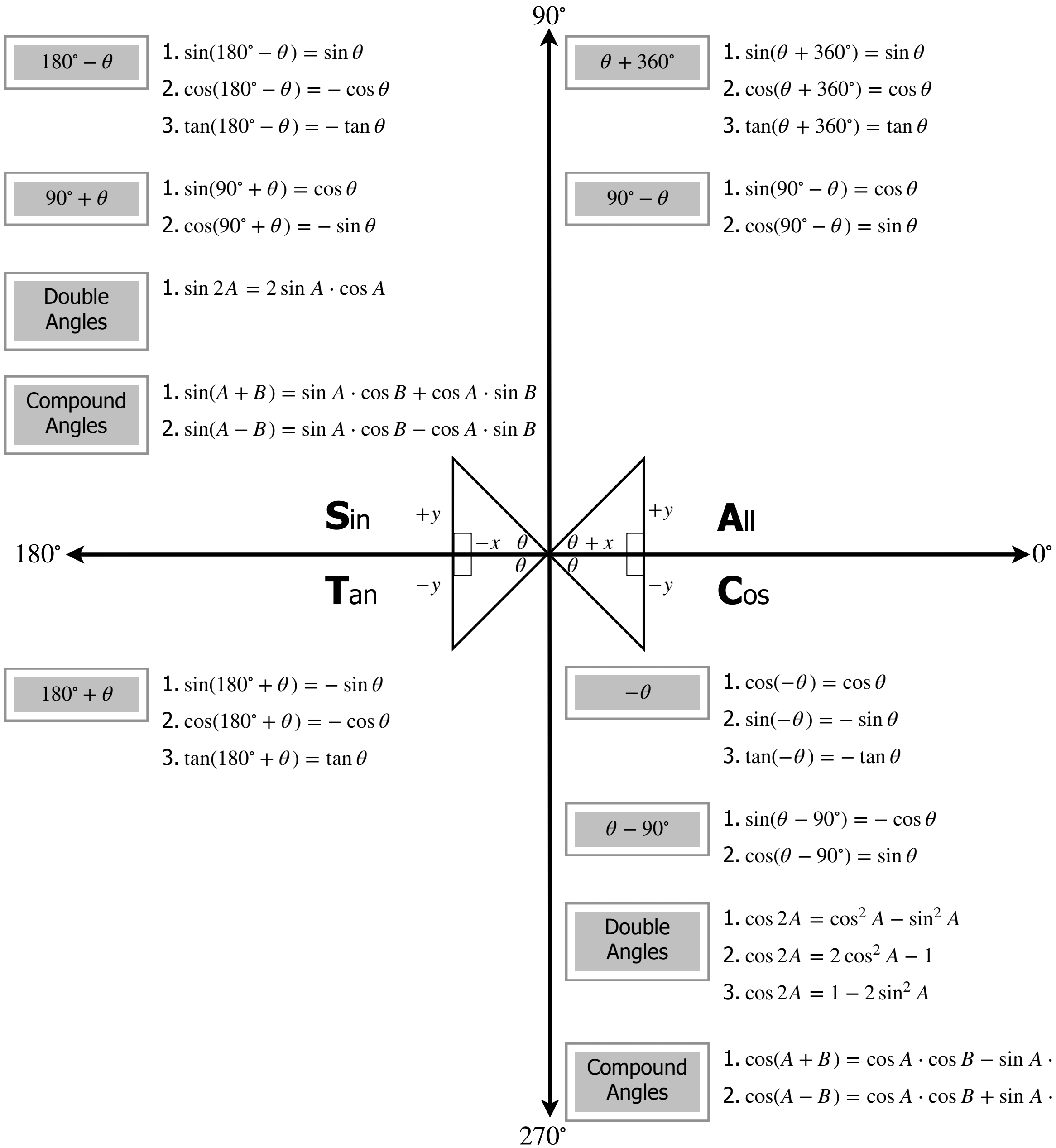
1. $\sin(90° - \theta) = \cos\theta$
2. $\cos(90° - \theta) = \sin\theta$

Double Angles

1. $\sin 2A = 2\sin A \cdot \cos A$

Compound Angles

1. $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$
2. $\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$



$180° + \theta$

1. $\sin(180° + \theta) = -\sin\theta$
2. $\cos(180° + \theta) = -\cos\theta$
3. $\tan(180° + \theta) = \tan\theta$

$-\theta$

1. $\cos(-\theta) = \cos\theta$
2. $\sin(-\theta) = -\sin\theta$
3. $\tan(-\theta) = -\tan\theta$

$\theta - 90°$

1. $\sin(\theta - 90°) = -\cos\theta$
2. $\cos(\theta - 90°) = \sin\theta$

Double Angles

1. $\cos 2A = \cos^2 A - \sin^2 A$
2. $\cos 2A = 2\cos^2 A - 1$
3. $\cos 2A = 1 - 2\sin^2 A$

Compound Angles

1. $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
2. $\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

GENERAL SOLUTIONS

- $\sin A = \sin B$ → ref angle
  - $A = B + k.360°$ or A
  - $A = 180° - B + k.360°;\ k \in \mathbb{Z}$
- $\cos A = \cos B$
  - $A = \pm B + k.360°;\ k \in \mathbb{Z}$
- $\tan A = \tan B$
  - $A = B + k.360°;\ k \in \mathbb{Z}$
- Other/ Co-function
  - $\sin A = \cos B\ \ \therefore \sin A = \sin(90° - B)$ → ref angle
  - $\cos A = \sin B\ \ \therefore \cos A = \cos(90° - B)$ → ref angle

EXAMPLE 1

Solve for x if $\cos 2x = -\sin x$

$1 - 2\sin^2 x = -\sin x$

$0 = 2\sin^2 x - \sin x - 1$

$0 = (\sin x - 1)(2\sin x + 1)$

$\sin x = 1$

$x = 0° + k.360°$

or

$x = 180° + k.360°;\ k \in \mathbb{Z}$

or

$\sin x = -\frac{1}{2}$

$x = -30° + k.360°$

or

$x = 210° + k.360°;\ k \in \mathbb{Z}$

EXAMPLE 2

Solve for x if $\cos(90° - x) \cdot \sin x - \cos 2x = 0$

$\sin x \cdot \sin x - (1 - 2\sin^2 x) = 0$

$\sin^2 x - 1 + 2\sin^2 x = 0$

$3\sin^2 x = 1$

$\sin^2 x = \frac{1}{3}$

$\sin x = \frac{\pm\sqrt{3}}{3}$

$x = 35{,}26° + k.360°$

or

$x = 180° - 35{,}26° + k.360°;\ k \in \mathbb{Z}$

$x = 144{,}74° + k.360°$

or

$x = -35{,}26° + k.360°$

or

$x = 180° - (-35{,}26°) + k.360°$

$x = 215{,}26° + k.360°$

TRIGONOMETRY

TRIG IDENTITIES

- $\tan x = \dfrac{\sin x}{\cos x}$
- $\dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$
- $\sin 2x = 2\sin x \cdot \cos x$
- $\sin 3x = \sin(2x + x)$

  $= \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$ (to be expanded further)
- $\sin 4x = \sin 2(2x)$

  $= 2\sin 2x \cdot \cos 2x$

  $= 4(\sin x \cdot \cos x)(\cos^2 x - \sin^2 x)$

  $= 4\sin x \cdot \cos^3 x - 4\sin^2 x \cdot \cos x$ (can be expanded further)
- $\cos 2x = \cos^2 x - \sin^2 x$

  $= 1 - 2\sin^2 x$

  $= 2\cos^2 x - 1$

HINTS FOR PROVING IDENTITIES

1. Start on the side with the least number of "terms" and simplify if possible.
2. Go to the other side and simplify until you get the same answer.
3. Look for a conjugate and multiply with the "opposite" sign (to make a difference of squares in the denominator of your fraction)
4. Always try to factorise where possible

EXAMPLE 1

Show\Prove that: $\dfrac{\sin 2x}{\cos 2x + \sin^2 x} = 2\tan x$

RHS $= 2 \cdot \dfrac{\sin x}{\cos x}$

LHS $= \dfrac{2\sin x \cdot \cos x}{(2\cos^2 x - 1) + (1 - \cos^2 x)}$

$= \dfrac{2\sin x \cdot \cos x}{\cos^2 x}$

$= \dfrac{2\sin x}{\cos x}$

$\therefore$ LHS = RHS

EXAMPLE 2

Show\Prove that: $\sin 3x = 3\sin x - 4\sin^3 x$

LHS $= \sin(2x + x)$

$= \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$

$= 2\sin x \cdot \cos x \cdot \cos x + (1 - \sin^2 x) \cdot \sin x$

$= 2\sin x \cdot \cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$\therefore$ LHS = RHS

EXAMPLE 3

Show\Prove that: $\dfrac{1 - \sin x}{1 + \sin x} = \left(\dfrac{1}{\cos x} - \tan x\right)^2$

LHS $= \dfrac{1 - \sin x}{1 + \sin x} \times \dfrac{1 - \sin x}{1 - \sin x}$

$= \dfrac{1 - 2\sin x + \sin^2 x}{1 - \sin^2 x}$

RHS $= \left(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right)^2$

$= \dfrac{1 - 2\sin x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1 - 2\sin x + \sin^2 x}{1 - \sin^2 x}$

$\therefore$ LHS = RHS

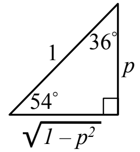
MIXED EXAMPLE 1

If $\sin 54° = p$, express the following in terms of p:

1. $\cos 36°$ 2. $\sin 108°$ 3. $\sin 84°$

Solutions:

$\sin 54° = \dfrac{p}{1}\ \left(\dfrac{o}{h}\right)$



1. $\cos 36° = p$
2. $\sin 108° = \sin 2(54°)$

   $= 2\sin 54° \cdot \cos 54°$

   $= 2(p) \cdot \left(\sqrt{1 - p^2}\right)$
3. $\sin 84° = \sin(54° + 30°)$

   $= \sin 54° \cdot \cos 30° + \cos 54° \cdot \sin 30°$

   $= p \cdot \dfrac{\sqrt{3}}{2} + \left(\sqrt{1 - p^2}\right)\left(\dfrac{1}{2}\right)$

   $= \dfrac{\sqrt{3}\,p + \sqrt{1 - p^2}}{2}$

MIXED EXAMPLE 2

Find the value of k if: $\cos 75° \cdot \sin 25° - \sin 75° \cdot \sin k = \sin 50°$

$\cos 75° \cdot \sin 25° - \sin 75° \cdot \sin k = \sin 50°$

$\cos 75° \cdot \sin 25° - \sin 75° \cdot \cos(90° - k) = \sin 50°$

$\sin(75° - 25°) = \sin 50°$

$\therefore k = 65°$

MIXED EXAMPLE 3

Express the following in terms of p if $\cos 73° \cdot \cos 31° + \sin 73° \cdot \sin 31° = p$

1. $\cos^2 21° - \sin^2 21° + 7$ 2. $\sin 42°$

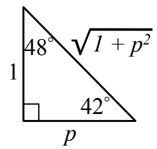
Solutions:

$\cos 73° \cdot \cos 31° + \sin 73° \cdot \sin 31°$

$= \cos(73° - 31°)$

$= \cos 42°$

$\therefore \cos 42° = \dfrac{p}{1}\ \left(\dfrac{a}{h}\right)$



1. $\cos 2(21°) + 7 = \cos 42° + 7$

   $= p + 7$
2. $\sin 42° = \dfrac{1}{\sqrt{1 + p^2}}$

QUESTION 1

.1 If $4\tan\theta = 3$ and $180° < \theta < 360°$, determine with the aid of a diagram:

.1.1 $\sin\theta + \cos\theta$ (4)

.1.2 tan 2θ (5)

.2 .2.1 Show that: $\dfrac{\cos(360° - x)\tan^2 x}{\sin(x - 180°)\cos(90° + x)} = \dfrac{1}{\cos x}$ (5)

.2.2 Hence, calculate without the use of a calculator, the value of:

$\dfrac{\cos 330° \tan^2 30°}{\sin(-150°)\cos 120°}$ (Leave your answer in surd form.) (2)

[16]

QUESTION 2

2.1 If sin 36° cos 12° = p and cos 36° sin 12° = q, determine in terms of p and q the value of:

2.1.1 sin 48° (3)

2.1.2 sin 24° (3)

2.1.3 cos 24° (3)

2.2 Show that $\sin^2 20° + \sin^2 40° + \sin^2 80° = \dfrac{3}{2}$

(**HINT**: 40° = 60° – 20° and 80° = 60°+ 20°) (7)

2.3 2.3.1 Prove: $\dfrac{\sin^4 x + \sin^2 x\cos^2 x}{1 + \cos x} = 1 - \cos x$ (4)

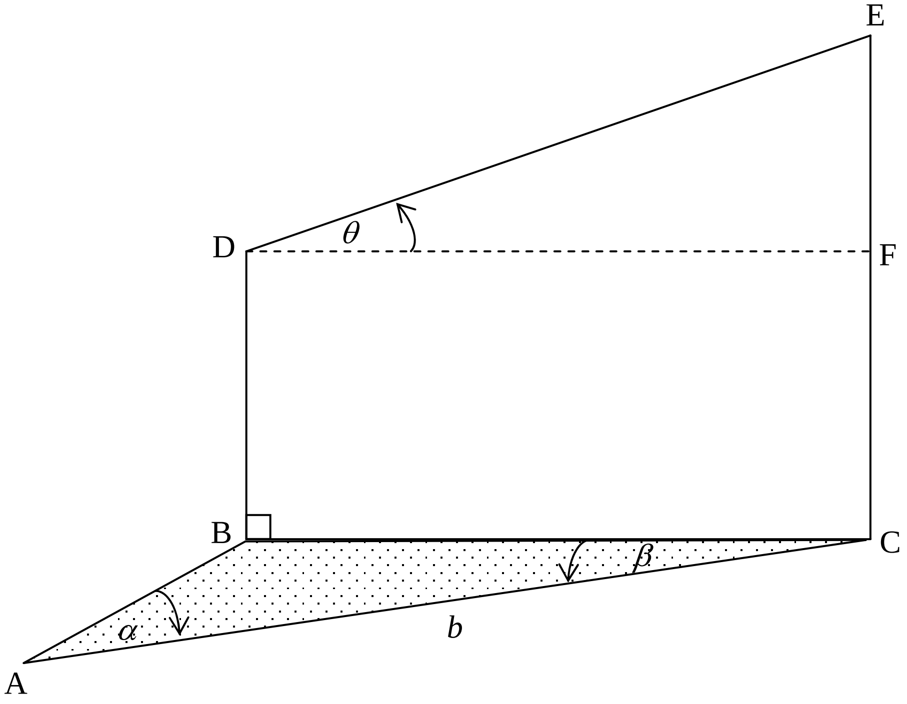
2.3.2 For which values of x is $\dfrac{\sin^4 x + \sin^2 x\cos^2 x}{1 + \cos x} = 1 - \cos x$ NOT true? (2)

[22]

QUESTION 3

In the diagram below A, B and C are three points in the same horizontal plane. D is vertically above B and E is vertically above C. The angle of elevation of E from D is $\theta°$. F is a point on EC such that DF || BC.

$B\hat{A}C = \alpha$, $A\hat{C}B = \beta$ and AC = b metres.



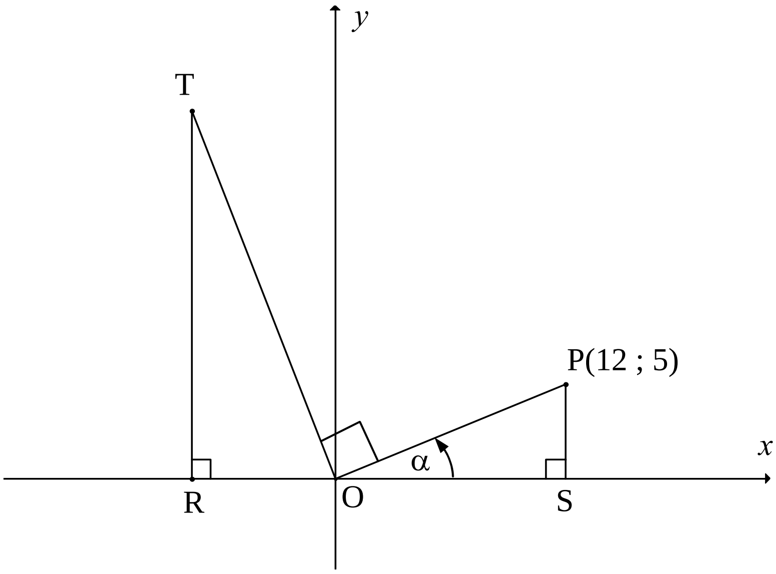
.1 Prove that $DE = \dfrac{b \sin\alpha}{\sin(\alpha + \beta)\cos\theta}$ (6)

.2 Calculate DE if b = 2 000 metres, $\alpha = 43°$, $\beta = 36°$ and $\theta = 27°$. (3)

QUESTION 4

.1 **Answer this question without using a calculator.**

In the diagram, P is the point (12 ; 5). OT ⊥ OP. PS and TR are perpendicular to the x-axis. $\hat{POS} = \alpha$ and OR = 7,5 units.



Determine:

.1.1 cos α (2)

.1.2 $\hat{TOR}$, in terms of α (2)

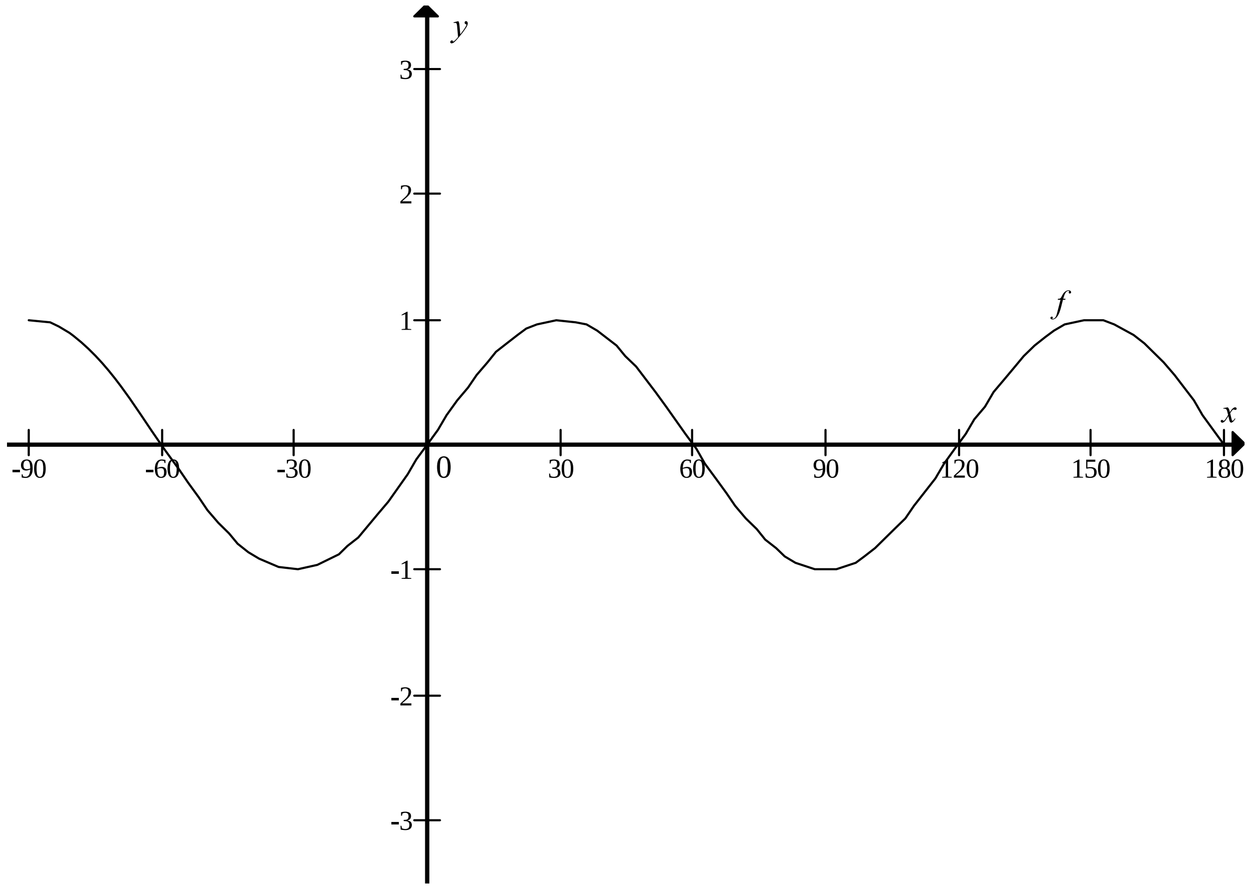
.1.3 The length of OT (4)

.2 Show that $\dfrac{\sin(90° + x).\cos x.\tan(-x)}{\cos(180° + x)} = \sin x$. (4)

[12]

QUESTION 5

The graph of $f(x) = \sin 3x$ is drawn below for $x \in [-90°\ ; 180°]$.



.1 Write down the period of f. (1)

.2 Write down the solutions for $\sin 3x = -1$ on the interval $[-90°\ ; 180°]$. (2)

.3 Give the maximum value of h if $h(x) = f(x) - 1$. (2)

.4 Draw the graph of $g(x) = 3\cos x$ for $x \in [-90°\ ; 180°]$ (3)

.5 Use the graphs to determine how many solutions there are to the equation $\dfrac{\sin 3x}{3} - \cos x = 0$ on the interval $[-90°\ ; 180°]$. (2)

.6 Use the graphs to solve: $f(x).g(x) < 0$. (4)

[14]

QUESTION 6

.1 If $\sin 61° = \sqrt{p}$, determine the following in terms of p:

.1.1 $\sin 241°$ (2)

.1.2 $\cos 61°$ (2)

.1.3 $\cos 122°$ (3)

.1.4 $\cos 73° \cos 15° + \sin 73° \sin 15°$ (3)

.2 .2.1 Prove the identity:

$$\frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x$$ (6)

.2.2 Determine a value of x in the interval [0° ; 180°] for which the identity is not valid. (2)

.3 .3.1 Given: $\sin x = \cos 2x - 1$. Show that $2\sin^2 x + \sin x = 0$. (1)

.3.2 Determine the general solution of the equation: $\sin x = \cos 2x - 1$. (6)

.4 Determine the value of:

$$\tan 1° \times \tan 2° \times \tan 3° \times \tan 4° \times ... \times \tan 87° \times \tan 88° \times \tan 89°.$$ (4)

[29]

QUESTION 7

Given: $f(x) = 1 + \sin x$ and $g(x) = \cos 2x$

.1 Calculate the points of intersection of the graphs f and g for $x \in [180° ; 360°]$. (7)

.2 Draw sketch graphs of f and g for $x \in [180° ; 360°]$ on the same system of axes provided on DIAGRAM SHEET 3. (4)

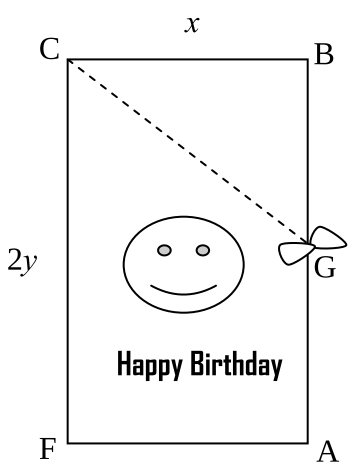
.3 For which values of x will $f(x) \le g(x)$ for $x \in [180° ; 360°]$? (3)

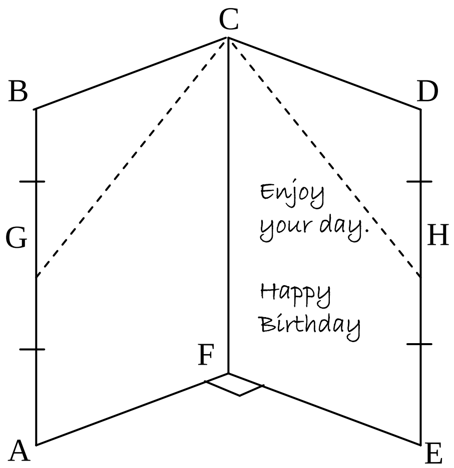
[14]

QUESTION 8

A rectangular birthday card is tied with a ribbon at the midpoints, G and H, of the longer sides. The card is opened to read the message inside and then placed on a table in such a way that the angle $A\hat{F}E$ between the front cover and the back cover of the card is $90°$. The points G and H are joined by straight lines to the point C inside the card, as shown in the sketch.

Let the shorter side of the card, BC = x, and the longer side, CF = $2y$.





Prove that $\cos G\hat{C}H = \dfrac{y^2}{x^2 + y^2}$. **[8]**

QUESTION 9

9.1 Simplify as far as possible: $1 \quad \sin^2 \qquad 3 \quad \cos^2$ (2)

9.2 Simplify WITHOUT the use of a calculator: $\sqrt{4^{\sin 150} \qquad 2^{3\tan 225}}$ (4)

9.3 Prove that $\dfrac{\cos^2 x \sin^2 x \quad \cos^4 x}{1 \quad \sin x} \quad 1 \quad \sin x$ (4)

9.4 Prove that for any angle , $\cos 3 \qquad 4\cos^3 \qquad 3\cos$.

(Hint: $3 \qquad 2$) (4)

9.5 If $x \quad \cos 20$, use QUESTION 9.4 to show that $8x^3 \quad 6x \quad 1 \quad 0$. (2)

[16]

QUESTION 10

.1 Simplify to ONE trigonometric function WITHOUT using a calculator:

$$\frac{\cos 160 \quad \tan 200}{2\sin(\quad 10 \ )}$$ (6)

.2 Consider $\cos(x \quad 45\ )\cos(x \quad 45\ )$.

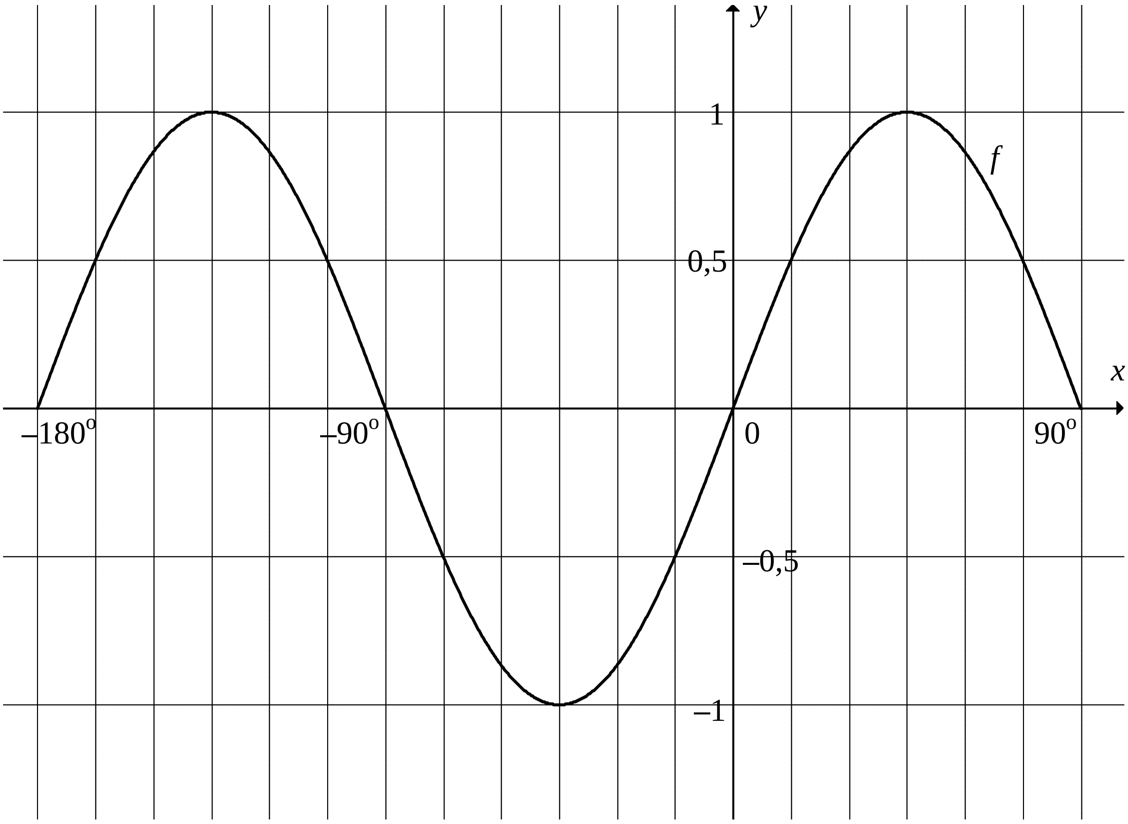
.2.1 Show that $\cos(x \quad 45\ )\cos(x \quad 45\ ) \quad \dfrac{1}{2}\cos 2x$. (4)

.2.2 Hence, determine a value of x in the interval $0 \quad x \quad 180$ for which $\cos(x \quad 45\ )\cos(x \quad 45\ )$ is a minimum. (3)

[13]

QUESTION 11

The graph of $f(x) = \sin 2x$ for $180 \quad x \quad 90$ is shown in the sketch below.



1 .1 Write down the range of f. (2)

1 .2 Determine the period of $f \quad \frac{3}{2}x$. (2)

1 .3 Draw the graph of $g(x) = \cos(x - 30°)$ for $180 \quad x \quad 90$. Clearly label ALL x-intercepts and turning points. (4)

1 .4 Hence, or otherwise, determine the values of x in the interval $180 \quad x \quad 90$ for which $f(x) . g(x) < 0$. (4)

1 .5 Describe the transformation that graph f has to undergo to form $y = \sin(2x + 60°)$. (2)

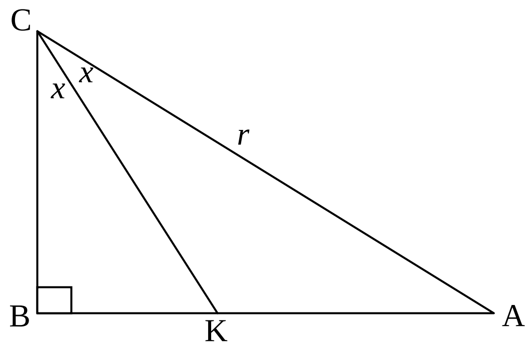
1 .6 Determine the general solution of $\sin 2x = \cos(x - 30°)$. (6)

[20]

QUESTION 12

In the diagram below, ABC is a right-angled triangle. KC is the bisector of $A\hat{C}B$.

AC = r units and $B\hat{C}K \quad x$.



12.1 Write down AB in terms of x and r. (2)

12.2 Give the size of $A\hat{K}C$ in terms of x. (1)

12.3 If it is given that $\frac{AK}{AB} \quad \frac{2}{3}$, calculate the value of x. (8)